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in it. Some might hold that its introduction into a university could not be justified unless it offered problems worthy of a Doctor's thesis. I believe that it does offer such problems, and as one such, mention the theory of illumination. But even if descriptive geometry did not offer such problems, its introduction into the college, and into the university, is justifiable because of the many reasons already given.

It is a fact that the authors of our textbooks on mathematics are deplorably ignorant concerning the making of drawings to represent space objects. Klein evidently had this in mind when he said, "Is it not as worthy an object of mathematics to be able to draw correctly as to be able correctly to calculate?" It seems only reasonable to demand that the drawings in mathematical textbooks should be made by some one who understands the underlying principles, and it would be very desirable for the author himself to be able to make his drawings.

As to the qualifications of a teacher of descriptive geometry, it would seem that he should have besides an extensive knowledge of the subject, covering the various branches above outlined, a knowledge also of plane and solid analytic geometry and projective geometry, besides the ability to make good drawings in ink as well as in pencil. The time is at hand when it is desirable to require of prospective teachers the completion of such a course. A person with such qualifications could be attached to either a department of mathematics or a department of drawing. If to the former, he would certainly be required to know more mathematics than the minimum just laid down, and if to the latter, he would probably be required to be proficient in other branches of drawing.

While this address was being prepared for publication, death claimed that one person who by many was regarded as the best teacher of descriptive geometry which this country ever had. I had the good fortune to be his pupil, and it was he who inspired me with that love for the subject which it is my earnest desire to transmit to others. I refer to the late Dr. E. A. Engler, at one time professor of mathematics at Washington University.

GEOMETRY FOR JUNIORS AND SENIORS.[1]

By E. B. STOUFFER, University of Kansas.

An examination of the list of papers presented to the American Mathematical Society during the last five years shows that on the average not more than one paper in six is concerned essentially with geometry. If the papers presented to the Society are assumed to picture approximately the research activity of America, it may well be asked whether geometry is receiving the attention which it deserves. Is a ratio of one to five or six a healthy sign for mathematics as a whole in this country?

It is true, there is no fundamental distinction between geometry and analysis—

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[1] Read at the second summer meeting of the Association, Cleveland, Ohio, September 7, 1917.

the laws which govern the process of reasoning in the two are the same. Yet frequently a theorem which requires a long and careful argument for its proof by the methods of analysis is intuitively evident when viewed geometrically. To see a proposition thus standing out by itself, stripped of its attendant proofs, is very much of an aid in grasping its full significance. Try, for instance, to explain to a class of sophomores the fundamental theorem of the integral calculus without showing that the integral may be represented as the area under a curve. Yet the proof is purely analytical.

The development of many a branch of mathematics may be aided by the development of geometry along some particular line. It cannot be denied, for instance, that the geometry of space of more than three dimensions has been a distinct asset to analysis where more than three variables are involved. The terms " points," " curves," " surfaces," etc., the notion of other elements than points as generating elements of geometrical figures, the propositions analogous to those in ordinary space, these may well suggest analytical theorems, otherwise unexpected, and may also greatly simplify their statement and proof. Thus, the problem of the equipartition of energy in a gas seems far removed from geometry, yet the notion of a sphere of n dimensions is used in the proof of some of the results already obtained towards its solution.

If, therefore, geometry has not kept pace in the scientific advance of America, it is pertinent to ask the cause and the cure; if it has maintained its proper position, it is still desirable to seek methods of insuring that, in the future, it does not retard the scientific world but rather leads the way.

The tendency to give geometry a secondary place is evident in many American institutions, both in the graduate and in the undergraduate courses. A man who received his master's degree in mathematics a year ago at one of our most important graduate institutions had had no geometry in advance of the ordinary plane and solid analytic geometry. Moreover, neither the student nor the institution is so very exceptional. As for the undergraduate student majoring in mathematics, he is required to take the traditional course in plane analytic geometry, and may be advised to take a brief course in solid analytic geometry, but there the list of courses in geometry open to him very frequently ends. He has no chance to gain the inspiration and delight which accompany a knowledge of the theorems bearing the names of Desargues, Pascal, Brianchon, no opportunity to recognize the geometry already studied as a special case of something far more general and more beautiful.

There can be no question but that many additional keen-minded students might become interested in mathematics if geometry were presented to them more attractively in their undergraduate courses. Sir G. H. Darwin, in the opening address of the Fifth International Congress, expressed the belief that analysis and geometry offer different attractions to mathematical minds and then added, " I suspect that the mathematician will drift naturally to one branch or another of our science according to the texture of his mind and the nature of the mechanism by which he works."

A fundamental cause then of our slow growth geometrically is our undergraduate course of study. The remedy, naturally, lies in the introduction of additional geometry of the proper kind, in suitable amount and in the best way. The questions before us are: What is the proper kind of geometry? What is a suitable amount? What is the best way of presenting it?

We assume the prerequisites of the ordinary freshman and sophomore mathematics: algebra, trigonometry, plane analytic geometry, differential and integral calculus. The student may have had additional courses—for example, one in synthetic geometry, in his freshman year. If so, it is his good fortune, but such a course should not replace one taken after the student's mind has been developed and broadened by the analytic geometry and the calculus.

There are two particular classes of students for whom the course should be planned: first, those who expect to teach mathematics in the secondary schools; and second, those who expect to continue their study of mathematics in a graduate school. There may be other students in the course—for instance, those who take the course for its so-called cultural value—but their number will probably be too small to require special attention. Besides, a course fulfilling the needs of the first two groups will undoubtedly be satisfactory to the stray student.

By no means all American institutions offer a course in geometry for juniors and seniors, but where there is such a course it is as a rule largely projective geometry. It might well be asked, however, whether there is not some other branch of geometry which would fit the needs of the class of students mentioned above better than the projective. In this paper attention will be called to some general principles which are believed to be fundamental in the selection and arrangement of the material for the course, regardless of whether it is projective geometry or not. If some other branch satisfies these requirements best, it should be given. However, in the discussion of these principles almost exclusive use has been made of the notions of projective geometry because, in the author's belief, no other branch of geometry can so well fulfill the conditions.

I. *The course should couple together the earlier courses in geometry, and should also form a connecting link between them and future courses if such are taken.*

There has been too much of a tendency to separate mathematics, especially for the beginner, into watertight compartments, each compartment containing the material for one course. The student frequently regards it as more important that he recognize the course in which a particular theorem has been presented than that he understand the theorem itself and its relation to other results. Most of us, I presume, have met the student who, when asked to solve a quadratic or simplify a complex fraction in trigonometry, explains that that is algebra and expresses surprise that such questions should be asked in a trigonometry class. That the unfortunate effects of this method of presenting the elementary courses have been recognized is evidenced by the number of texts which have recently appeared combining in various ways the algebra, trigonometry, analytic geometry and calculus. This question was even considered worthy of formal discussion at the summer meeting of the Association a year ago.

It is equally important that the student recognize that the traditional divisions of *geometry* are for the purpose of emphasizing some unique method of attack or some particular kind of results. The high school work in geometry is entirely synthetic and is largely a course in memory work and logical reasoning; the early college courses, on the other hand, are as a rule concerned with teaching the "analytic method," that is, with showing the combined use of coördinates and algebra. It is true that the student recognizes some of the terms, like point, line, circle, as common, and even a few identical theorems may be proved in both sets of courses; but the methods employed seem to have nothing in common.

This evil is especially serious for one who goes out to teach mathematics in the high school without further training in geometry. He may have understood fully his college course in analytic geometry but he can only teach his Euclid in the way he first learned it, with no inspiration for himself and little for his students. Klein calls this jump from the synthetic geometry of the high school to the analytic geometry and back again a "double discontinuity" and adds that the real purpose of a university course for the teacher of mathematics is "that you may be able in a large measure to draw inspiration for your teaching from the great body of knowledge that has been presented to you."

II. *The course should be both synthetic and analytic, with the emphasis on the synthetic.*

After Poncelet and his followers, in the early part of the nineteenth century, had shown the possibilities of synthetic work in projective geometry, the wave of enthusiasm for it threatened for a time to drive the analytic school out of existence. Although the brilliant work of von Staudt in freeing synthetic geometry from analytic and metric relations was a triumph for the synthetic school, the later work of Plücker and Cayley showed that each method had its distinct advantages and that it was a decided loss to cultivate one at the expense of the other.

The entire elimination of the analytic work from the first course in projective geometry is especially unfortunate for the student who has just finished a course in analytic geometry. For, it is only by the introduction of some analytic work that the two courses can be connected in a vital way. This fact is well expressed by Segre in a strong appeal for combined analytic and synthetic methods in geometry: "The method of coördinates serves to pass from one to the other and unites them intimately, or rather so welds them together that we may say that every advance in the one means an advance in the other."[1]

It is not desirable to introduce a large proportion of analytic work, as the student is already much better trained in its methods than in synthetic methods. Results which can be obtained equally well or better by the synthetic method should be presented in that way. However, even a relatively small amount of analytic work in the projective geometry will strengthen and enlarge both the elementary analytic geometry and the projective geometry. For instance, the student who has had only the usual course in analytic geometry feels no freedom in the use of coördinate systems. The idea that other quantities than distances—for example, the ratio of distances or the value of an angle—may be used as coördinates, the notion that lines and planes can be determined by coördinates in place of equations, the concept of homogeneous coördinates and the resultant elimination of the infinitely distant point as a special case, all these are new viewpoints which broaden previous concepts remarkably.

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[1] Segre, *Rivista di Matematica*, Vol. I (1891), pp. 42–66; translation by J. W. Young, *Bulletin of the American Mathematical Society*, Series II, Vol. 1, pp. 442–468.

On the other hand, the projective geometry gains decidedly from the elementary analytic geometry. The latter assumes implicitly that there is a one-to-one correspondence between the points on a line and the numbers of the real number system. The real basis for the assumption is admittedly too difficult for inclusion in an undergraduate course, but the same may be said of the alternative assumption, the postulate of continuity in some form or other. The introduction of analytic methods into the projective geometry makes possible the explicit assumption of an abscissa system of coördinates for the points of a range, an assumption already familiar to the student and in fact quite natural and easy of comprehension. From this assumption follows directly the projective coördinate system and the proof of the fundamental theorem of projective geometry.

Again, in the introduction of imaginaries into geometry, analysis comes to the rescue. If statements are to be made as general as possible—and that is one of the principal features of this course—imaginary elements must be considered. Von Staudt's use of involutions without double points for defining imaginary points on a line, while a fine piece of mathematical work, is too difficult for such a course as we are outlining. By the use of algebra the whole subject of imaginaries is carried back to a field with which the student is already somewhat familiar. The necessity of treating imaginaries in geometry by means of algebraical quantities is well expressed by Russell in his *Foundations of Geometry*, where he says: "All the fruitful uses of imaginaries in geometry are those which begin and end with real quantities, and use imaginaries only for the intermediate steps. Now in all such cases, we have a real spatial interpretation at the beginning and end of our argument, where alone the spatial interpretation is important; in the intermediate links, we are dealing in a purely algebraical manner with purely algebraical quantities and may perform any operations which are algebraically permissible."

The possibility of giving an analytical statement to the projective transformation is another gain from the introduction of analytic methods. The equation of the transformation not only makes the whole transformation appear much more real but affords a solid peg on which to hang its double points. Moreover, the slight acquaintance here with the linear transformation will make the friendship ripen much more rapidly when it is met in some other course in geometry, in function theory, or elsewhere.

The prospective graduate student in mathematics will take this course as a preparation for the more intensive graduate courses. He should therefore have training in both analytic and synthetic methods. "Often it will be found convenient to alternate between the synthetic method which appears more penetrating, more enlightening, and the analytic which is in many cases more powerful, more general, or more rigorous; and it will frequently happen that the same subject will not be quite clear from all points of view unless treated by both methods. . . . The most important thing is that the geometer should not be the slave of a single method and that he should put himself in a position where he can make use of any instrument to obtain an important result."[1]

III. *The course should begin with undefined terms and unproved propositions, specifically stated.*

"Foundations" is too large a word for what should be included, but there must always be some undefined terms and some unproved propositions. These should be as simple as possible. The course might begin with about the kind of material Enriques uses in the first three sections of his projective geometry. The point, line, and plane are assumed without definition, the nine fundamental forms are then defined, and eight or ten propositions from elementary geometry, so simple as to appear intuitively correct, are introduced without proof. Also the assumptions are made which are necessary for the elimination of the special cases of infinite elements.

The question of foundations is extremely important for the prospective teacher of Euclidean geometry, not so much because he will present that question to his students as because his own clear ideas will enable him to judge what to emphasize and what to omit. It is either an unusually brilliant or an exceedingly careless teacher who has never been troubled with logical difficulties in the treatment of certain topics in algebra and geometry. The emphasis given to logical considerations in the report of the National Committee of Fifteen on Geometry Syllabus is sufficient evidence of the importance of this subject in elementary geometry; but it must be difficult for the teacher to grasp the significance of this report if the idea of making necessary assumptions and deducing results therefrom has not been presented to him in some form or other in college courses.

IV. *The course should show the generalizing methods and principles which have been so characteristic of modern work in geometry.*

This should undoubtedly be the chief aim of the course. The unfolding to the interested student of the geometrical concepts developed during the early part of the nineteenth century will cause him to experience in his own mind much the same transformation as that through which the mathematical world passed during those years. In fact, there is a noticeable parallelism between the development of mathematics through the centuries and its development in the mind of the student under the modern curriculum. There is in both first arithmetic, then algebra and Euclidean geometry, followed in order by Cartesian geometry and the calculus.

Historically the next great advance opened up the field of the so-called modern geometry. The developments of the eighteenth century had been almost entirely analytic. The newly discovered calculus and its unlimited applications, with the assistance of the analytic geometry, had crowded out of consideration any other forms of geometry. But by the end of the eighteenth century this field of research was beginning to appear exhausted and there was a revival of interest in the almost forgotten work of Desargues and Pascal. The French mathematician Monge really reopened the subject of projective geometry by the publication in 1800 of his *Descriptive Geometry.* Besides discovering many properties of curves and surfaces useful for classification purposes, he first suggested the use of imaginaries in pure geometry. His work introduces into geometry, as Hankel says, " the hitherto unknown idea of geometric generality and geometric elegance."

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[1] Segre, *loc. cit.*

In 1822 there appeared a treatise by Poncelet, a pupil of Monge, in which is introduced for the first time the valuable concept of figures in homology. Here are discussed also such notions as the line at infinity, the circular points at infinity, imaginaries, and projection and section. Plücker and Steiner soon followed with treatises, that of Steiner being especially worthy of notice because in it projective pencils are first made use of in the generation of conics. " In the ten years which embrace the publication of these immortal works of Poncelet, Plücker and Steiner, geometry has made more real progress than in the two thousand years which had elapsed since the time of Apollonius. The ideas which had been slowly taking shape since the time of Descartes suddenly crystallized and almost overwhelmed geometry with an abundance of new ideas and principles."[1]

If the mathematical world was so inspired by the richness and beauty of the projective geometry after a period devoted exclusively to analytic geometry and the calculus, is it not to be expected that there will be a favorable reaction in the mind of the college student when, after the two years spent on the usual freshman and sophomore courses, he sees the generality of the projective methods? Duality will appeal to him not merely as a great labor-saving device, but also as a new and unexpected correlation of widely separated theorems. Nothing of the kind has appeared to him in previous courses. Likewise the process of projection and section is both easy to understand and fascinating in its possibilities. Again, the elimination of the consideration of the infinitely distant element as a special case—the horror of the average student—brings undiluted delight. The invariance of the anharmonic ratio under projective transformations, the possibility of generating figures by means of other elements than points, the notion of coordinates attached to lines and planes, the construction of conics by means of two projective pencils, these and many other such general notions cannot fail to astonish and enthuse a student who has any interest at all in geometry.

V. *Metric geometry should be introduced as a special case, but emphasis should be put on the fact that it is metric.*

The introduction of metrical considerations brings the whole subject back to actual experience. The possibilities of the applications of projective geometry to physical problems and the presentation of metric geometry as a special case of a more general geometry are immediate results. Moreover, in no other way can the important part the infinitely distant point plays be so easily emphasized.

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[1] Pierpont, "History of Mathematics in the Nineteenth Century," *Bulletin of the American Mathematical Society,* Series II, Vol. 11, pp. 136–159.

There might be mentioned several other general principles which should be considered in the selection and arrangement of the course. For instance, the material should be arranged as far as possible in the order in which the student would of himself seek information; there should be an abundance of construction problems; the problems of the course should not only afford drill in the theory but should also offer chance for originality without being so difficult as to produce discouragement.

To leave these somewhat safer generalities, what should be the content and arrangement of topics in order to fulfill best the requirements mentioned? A very brief outline of a plan for a course is here suggested.

After the undefined elements and unproved propositions from elementary geometry have been introduced and the method of eliminating the consideration of the ideal elements as special cases has been explained, the principle of duality may be explained and illustrated. We are then ready for a discussion of projection and section, and of perspectivity, and for the definition of a projectivity as a sequence of perspectivities. Desargues's theorem on triangles should be proved as an illustration of the possibilities of the use of projectivities.

We are now ready for the introduction of coördinate systems for one-dimensional forms. Emphasis should be put on the fact that we seek a system satisfactory for projective geometry, that is, such a system that the coördinate of an element need not be changed when the form is subjected to a projective transformation. The abscissa system, introduced by means of the assumption already mentioned, and the simple ratio system, which follows immediately, both fail to satisfy the requirements of projective geometry. The double-ratio coördinate follows most naturally and is found to be invariant under a projectivity. As already mentioned, it is now easy to prove the fundamental theorem of projective geometry.

The projective transformation may next be shown to be equivalent to the linear transformation. The double points of a projectivity appear at once. Some metric geometry may be introduced and also a number of interesting construction problems. If time permits a discussion of geometric addition and multiplication will be found illuminating.

Harmonic sets with some metric properties should now be introduced. Involutions may follow, being defined most directly perhaps as projectivities of period two. The analytic representation of an involution makes intelligent a discussion of its double elements, even though they are conjugate imaginaries.

We are now ready for the discussion of conics. The field of selection here is so large and varied that the time at the disposal of the class must determine what is given, but certainly there should be included the famous theorems of Pascal, Brianchon and Desargues, projectivities on a conic, the theory of poles and polars, and some metrical properties of conics.

Conics might be either preceded or followed by a discussion of two-dimensional projective coordinates. The duality between points and lines, both as to coördinates and equations, the use of the abridged notation, the expression of the double ratio of four elements in terms of their parameters, the equations of conics, are all worthy of particular notice. Any time remaining might well be given to further discussion of projectivities between two-dimensional forms.

The course as outlined will probably consume the time of a three-hour course throughout the year. If only half that amount of time can be given to the course, as is unfortunately true in many institutions, it will be necessary to omit the work on two-dimensional coordinate systems and to make brief the treatment of some of the other subjects mentioned. In any case, however, the course is such that the prospective teacher will have new inspiration as well as much new material, and that the future graduate student will have an increased interest in geometry and a first-class foundation for its further study.

BOOK REVIEWS AND NOTES.

SEND ALL COMMUNICATIONS TO W. H. BUSSEY, University of Minnesota.

BOOK REVIEWS.

An Introduction to Statistical Methods. By HORACE SECRIST, PH.D., Associate Professor of Economics and Statistics, Northwestern University. The Macmillan Company, New York, 1917. xxi + 482 pages. $2.00.

We are told in the preface to this book that "the treatment is non-mathematical for several reasons, chief of which are that the mathematical phases of the subject are treated in other places and that there seems to be an urgent need for a fundamental discussion of the non-mathematical but not less vital processes in statistical investigation and analysis."

With a non-mathematical treatment, we may well question the propriety of reviewing the book in a mathematical journal, but the methods involve certain mathematical concepts such as average deviation, standard deviation, correlation coefficient, and probable error.

It may well be said that it seems to the reviewer that the illustrations drawn from economics and business are a very useful and interesting part of the book. There is given a valuable discussion of the collection of statistical data with special reference to the dangers of interpreting results from figures collected without a thorough knowledge of the conditions and limitations under which the figures are produced. The main sources of economic and business statistics are also well discussed.

In the chapters on "Tabular Presentation," "Diagrammatic Presentation," and "Graphic Presentation," the author indicates how the choice of different devices depends largely on the purpose of the presentation.

In the chapter on "Averages as Types," the uses and abuses of different